MTH210 Summary Sheet Jeff Mair

5.1 Sequences:

- Separation of last term: $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n-1} a_i + a_n$
- Telescoping sum
- Properties of Summation and Product series
- Convert base 10 to base 2 by division
- Factorial manipulation

5.1 n choose r:

$$\binom{n}{r} = nCr = \frac{n!}{r!(n-r)!}$$

$$\sum_{i=m}^{n} a_k + \sum_{i=m}^{n} b_k = \sum_{i=m}^{n} (a_k + b_k)$$

$$\prod_{i=m}^{n} a_k \cdot \prod_{i=m}^{n} b_k = \prod_{i=m}^{n} (a_k \cdot b_k)$$

Induction II
$k^2 < 2^k \quad \forall k \geq 5$
$(k+1): k^2+2k+1 < 2^k + 2k+1$
But $2k+1 < 2^k$ for $k \geq 5$
So $(k+1)^2 < 2^k + 2^k$
etc

5.2 Mathematical induction:

1. Basis step. Show P(a) is true (choose 0 or 1 usually)
2. Inductive hypothesis. Suppose P(k) is true.
   [write the formula in terms of k]
3. Show that P(k+1) is true. Manipulate P(k+1) to a form where P(k) can be subbed in.

5.2 Sum of the First n Integers

$$1+2+3+4+...+n = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} a_i$$

Example: $3+4+5+6+...+1000 = ?$
$= 1+2+3+4+...+1000 - (1+2)$
$= \frac{1000(1001)}{2} - 3$

5.2 Sum of a Geometric Sequence

$$1+r+r^2+...+r^n = \frac{r^{n+1}-1}{r-1} \qquad \sum_{i=0}^{n} r^n$$

5.3 Mathematical Induction (weak)

- Remember properties of Logarithms and properties of inequalities!

k(k + 1) -> product of two consecutive intengers is always even

Typical case statements in proofs: k+1 is prime/not prime, k+1 is odd/even.

5.4 Strong Mathematical Induction

1. P(a), P(a+1),... P(b) are shown to be true.
2. For any int, $k \geq b$, if P(i) is true for ints $a \leq i \leq k$, then P(k+1) is true.

$\therefore$ for all ints, $n \geq a$, P(n).

5.6 Recursion

Arithmetic sequence:
$a_k = a_{k-1} + d$ OR $a_n = a_0 + n \cdot d$

Geometric sequence:
$a_k = r \cdot a_{k-1}$ OR $a_n = a_0 \cdot r^n$

Fibonacci: $f_n = f_{n-1} + f_{n-2}$ OR $f_{n-1} + f_n = f_{f+1}$
{1, 1, 2, 3, 5, 8, 13, 21, ...}

Tower of Hanoi: $M_k = 2 \cdot M_{k-1} + 1$

	Repeated edge	Repeated Vertex	Start/end Same point	must Contain at least 1 edge
walk	allow	allow	allow	N
trail	N	allow	allow	N
path	N	N	N	N
closed walk	allow	allow	Y	N
cct	N	allowed	Y	Y
Simple cct	N	[illegible]	Y	

5.7 Solving Recurrence Relations by Iteration

Arithmetic Sequence: $a_k = a_{k-1} + d \rightarrow a_n = a_o + d \cdot n$

Geometric Sequence: $a_k = r \cdot a_{k-1} \rightarrow a_n = a_o \cdot r^n$

5.9 General Recursive Definitions and Structural Induction

McCarthy's 91 Function

$$M(n) = \begin{cases} n - 10, & \text{if } n > 100 \\ M(M(n+11)), & \text{if } n \leq 100 \end{cases}$$

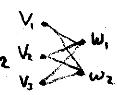
Ackerman's Function

A(0, n) = n + 1
A(m, 0) = A(m - 1, 1)
A(m, n) = A(m - 1, A(m, n - 1))

Recursive Function that is not "well defined"

10.1 Graphs Simple = no loops, or parallel edges

- Simple; Complete; K_n; Complete Bipartite $K_{(m,n)}$ $K_{3,2}$



Deg of each V is n-1; Total deg is n(n-1);

So, num of edges of K_n is $\frac{n(n-1)}{2}$

Total degree is always 2 x number of edges

Total degree is always even

Always an even number of verticies with odd degree

10.2 Trails, Paths, Circuits

Path, Trail (+v), Walk (+v, +e)

Simple cct, cct

Closed walk

Euler Cct

Every V at least once, every E once-only;
Starts/stops on same V;
No V has odd degree

Euler Theorem: if G is connected and the deg of every V is even (and > 0) then G has an Euler cct

Hamiltonian Cct

Simple cct that includes every V (does not repeat) of G;
Does not need to use every edge;
Start/stop on same V

Hamiltonian Proposition: If G has Hamil. cct, then G has a subgraph H with the following properties:

1. H contains every V of G.
2. H is connected.
3. H has same number of E as V
4. Every V of H has degree 2

10.4 Isomorphism

Invariant graph isomorphism properties:

1. has n vertices;
2. has m edges;
3. has a vertex of degree k;
4. has m vertices of degree k;
5. has a cct of length k;
6. has a simple cct k;
7. has m simple cct of length k;
8. is connected;
9. has an Euler cct;
10. has a Hamiltonian cct.

10.5 Trees

Def: cct free, and connected

For n verticies, a tree has n - 1 edges, so 2(n-1) degree in total

$32-9=23 \geq 9$, $32-9-9=14 \geq 9$, $32-9-9-9=5<9$
∴ 3 groups of 9's with 5 left over.

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4.8 Division Algorithm

Euclidian Algorithm for finding GCD

$$\gcd(a,b) = \gcd(b,r) \text{ st } \frac{a}{b} = q + \frac{r}{b}$$

Prime Factorization technique for finding GCD (min powers)

ab = gcd(a,b) x lcm(a,b)

8.4 Modular Arithmetic

$a \equiv b (mod\ m) \rightarrow m|(a-b) \rightarrow a = b + kn$

Properties:

$a \equiv c (mod\ n)$ relatively prime ⇔ $\gcd(a,b)=1$

$b \equiv d (mod\ n)$ -Must be R.P. to Find inverse modulo.

$(a+b) \equiv (c+d)(mod\ n)$

$(a-b) \equiv (c-d)(mod\ n)$ if a negative value is found sum with mod #.

$ab \equiv ad\ (mod\ n)$

$a^m \equiv c^m (mod\ n),\ \forall m \in Z$

- rewriting GCD as a linear combination
- find inverse modulo

$14^2 \bmod 55 = 196 \bmod 55 = 31$

$14^4 \bmod 55 = (14^2 \bmod 55)^2 \bmod 55 = 31^2 \bmod 55 = 961 \bmod 55 = 26$, etc

$14^{27} \bmod 55$; $27 = 2^4 + 2^3 + 2^1 + 2^0$

$= [(14^{16} \bmod 55) \cdot (14^8 \bmod 55) \cdot (14^2 \bmod 55) \cdot (14 \bmod 55)] \bmod 55 = 9$

$210 \cdot x \equiv 8 \pmod{13}$

1. Rewrite as a Linear Combination: $210(7) \equiv 1 \pmod{13}$
2. Mult by $8 \equiv 8 \pmod{13}$ ⇒ $210 \cdot 56 \equiv 8 \pmod{13}$

Additional Topics

Chinese Remainder Thm

Divisibility by N

9.1 - counting/probablility

Where we have equal likelyness:

$$P(E) = \frac{\text{Num of outcomes in Event E}}{\text{Total Num of Outcomes in S}}$$

Elements in a list = n - m + 1

9.2 Multiplication Rule

Number of combinations with/without repetition

Permutations: (aka arrangements)

Ordering of a set of objects in a row

For n unique items, there are n! permutations

Permutations of selected items in a set:

$$P(n,r) = \frac{n!}{(n-r)!}$$

# of integers from 10 to 99:

$[\text{\# of ways to pick First digit}] \cdot [\text{\# of ways to pick 2nd digit}] = [1,9][0,9]$
$= 9 \cdot 10$
$= 90$

9.3 addition rule

Logical "or" = $N(A \cup B) = N(A) + N(B)$

Difference rule

∘ Probability, $P(A^c) = 1 - P(A)$

Inclusion/Exclusion

$N(A \cup B \cup C) = N(A) + N(B) + N(C)$
$- N(A \cap B) - N(A \cap C) - N(B \cap C)$
$+ N(A \cap B \cap C)$

*multiplication

9.4 Pigeonhole; n objects, m containers

∘ at least 1 container contains $\geq \lceil n/m \rceil$

∘ at least 1 container contains $\leq \lfloor n/m \rfloor$.

∘ Common divisors greater than 1

9.5 r "combinations" ⇒ $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

How many distinguishable orderings of letters of MISSISSIP

$\binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1}$

CRT

$m = m_1 \cdot m_2 \cdot m_3$

$M_k = m/m_k$

$M_k y_k \equiv 1 \pmod{m_k}$

$X = a_1 M_1 y_1 + a_2 M_2 y_2 + a_3 M_3 y_3$

then

$X \equiv ? \pmod{m}$

then

"$X_{general} = ? + m \cdot k$" [find general sol'n]

are m_1, m_2, m_3 all R.P?

eg 6, 10, 15 → Not R.P.

$X \equiv 5 \pmod{6}$
$X \equiv 3 \pmod{10}$
$X \equiv 8 \pmod{15}$

Remainder

$X \equiv 5 \pmod{6}$

$\begin{cases} X \equiv 5 \pmod{2} \equiv 1 \pmod{2} \\ X \equiv 5 \pmod{3} \equiv 2 \pmod{3} \end{cases}$

etc...

Find distinct eq's.

$X \equiv 53 \pmod{30}$

$\Rightarrow \equiv 23 \pmod{30}$

(remainder must always be smaller!)